Problem 4.26

In Problem 4.4 you showed that

$$Y_2^1(\theta,\phi) = -\sqrt{15/8\pi}\,\sin\theta\cos\theta\,e^{i\phi}.$$

Apply the raising operator to find $Y_2^2(\theta,\phi)$. Use Equation 4.121 to get the normalization.

Solution

According to the result of Problem 4.21, applying the raising operator to a function gives

$$\begin{aligned} L_+ f_\ell^m &= A_\ell^m f_\ell^{m+1} \\ &= \hbar\sqrt{\ell(\ell+1)-m(m+1)}\, f_\ell^{m+1}. \end{aligned}$$

Apply the raising operator to $Y_2^1(\theta,\phi)$ to get $Y_2^2(\theta,\phi)$.

$$\begin{aligned} L_+ Y_2^1(\theta,\phi) &= \hbar\sqrt{2(3)-1(2)}\,Y_2^2(\theta,\phi) \\ &= 2\hbar Y_2^2(\theta,\phi) \end{aligned}$$

Therefore,

$$\begin{aligned} Y_2^2(\theta,\phi) &= \frac{1}{2\hbar}L_+ Y_2^1(\theta,\phi) \\ &= \frac{1}{2\hbar}\left[\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\right]\left(-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,e^{i\phi}\right) \\ &= -\frac{e^{i\phi}}{2}\sqrt{\frac{15}{8\pi}}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)(\sin\theta\cos\theta\,e^{i\phi}) \\ &= -\frac{e^{i\phi}}{2}\sqrt{\frac{15}{8\pi}}\left[\frac{\partial}{\partial\theta}(\sin\theta\cos\theta\,e^{i\phi}) + i\cot\theta\frac{\partial}{\partial\phi}(\sin\theta\cos\theta\,e^{i\phi})\right] \\ &= -\frac{e^{i\phi}}{2}\sqrt{\frac{15}{8\pi}}\left[e^{i\phi}\frac{d}{d\theta}(\sin\theta\cos\theta) + i\left(\frac{\cos\theta}{\sin\theta}\right)\sin\theta\cos\theta\frac{d}{d\phi}(e^{i\phi})\right] \\ &= -\frac{e^{i\phi}}{2}\sqrt{\frac{15}{8\pi}}\left[e^{i\phi}(\cancel{\cos^2\theta} - \sin^2\theta) + \cancel{i\cos^2\theta\,(ie^{i\phi})}\right] \\ &= -\frac{e^{i\phi}}{2}\sqrt{\frac{15}{8\pi}}(-e^{i\phi}\sin^2\theta) \\ &= \sqrt{\frac{15}{32\pi}}\sin^2\theta\,e^{2i\phi}. \end{aligned}$$